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Complementary bases in symplectic matrices and a proof that their determinant is one ☆

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Abstract

New results on the patterns of linearly independent rows and columns among the blocks of a symplectic matrix are presented. These results are combined with the block structure of the inverse of a symplectic matrix, together with some properties of Schur complements, to give a new and elementary proof that the determinant of any symplectic matrix is +1. The new proof is valid for any field. Information on the zero patterns compatible with the symplectic structure is also presented.

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1. Introduction

Let I_n denote the n-by-n identity matrix and J the $2n$-by-$2n$ matrix

$$J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

J is an orthogonal, skew-symmetric matrix, so that $J^{-1} = J^{\mathrm{T}} = -J$.

Definition 1.1. A $2n$-by-$2n$ matrix S with entries in the field $\mathbb{K}$ is called symplectic, or J-orthogonal, if $S^{\mathrm{T}}JS = J$.

The group of the symplectic matrices is a relevant class of matrices both from a pure mathematical point of view [9], and from the point of view of applications. For instance, the symplectic matrices play an important role in classical mechanics and Hamiltonian dynamical systems [1,2]. Moreover, the symplectic matrices appear in the linear control theory for discrete-time systems [4,10].

Notice that the symplectic matrices represent the isometries of the skew-symmetric bilinear form defined by J, i.e., $\langle x, y\rangle = x^{\mathrm{T}}Jy$, where x and y are $2n$-by-1 column vectors. If the field $\mathbb{K}$ is $\mathbb{C}$, one can also consider the sesquilinear form $\langle x, y\rangle = x^{*}Jy$, where x^{*} is the conjugate transpose of x. In this case, the isometries correspond to those matrices S such that $S^{*}JS = J$. These matrices are called *conjugate symplectic matrices*[1] in the literature [3,7]. We will not deal explicitly with conjugate symplectic matrices, however we will remark in the proper places that some of the presented results remain also valid for these matrices.

The following properties are very easily proved from Definition 1.1: the product of two symplectic matrices is also symplectic; if S is symplectic then S^{-1} and S^{T} are symplectic; and $\lambda \in \mathbb{K}$ is an eigenvalue of S if and only if $\lambda^{-1} \in \mathbb{K}$ is an eigenvalue of S.

Definition 1.1 and the multiplicativity of the determinant imply directly that $\det S = \pm 1$. *However, the determinant of a symplectic matrix is, actually, always* +1, for any field $\mathbb{K}$. If the elements of the field $\mathbb{K}$ can be considered complex numbers (for instance if $\mathbb{K}$ is the field of real numbers or the field of rational numbers), the property $\det S = +1$ is equivalent to the fact that if -1 is an eigenvalue of a symplectic matrix then its algebraic multiplicity is even. Also, it is equivalent to the fact that if +1 is an eigenvalue of a symplectic matrix then its algebraic multiplicity is also even. These two properties can be seen as follows: notice that $S = J^{-1}S^{-\mathrm{T}}J$. This means that if $\lambda \neq \lambda^{-1}$ is an eigenvalue of S, then λ^{-1} is also an eigenvalue of S with the same algebraic multiplicity as λ. Taking into account that the dimension of S is even, the sum of the algebraic multiplicities of +1 and -1 (if some of them is an eigenvalue) is even. Moreover $\det(S) = (-1)^{a}$, where a is the algebraic multiplicity of -1.

It has long been known that the determinant of any symplectic matrix is +1, but no proof seems entirely elementary. This has been recognized by several authors. Let us quote, for instance, the following paragraph from [9]:

> It is somewhat nontrivial to prove that the determinant itself is +1, and we will accomplish this by expressing the condition for a matrix to be symplectic in terms of a differential form … Another proof may be found in Arnold [2].

[1] The matrices S satisfying $S^{*}JS = J$ and having complex entries are also frequently called in the literature complex symplectic matrices [6]. We do not use this name, because it may produce confusion when compared with Definition 1.1 in the case $\mathbb{K} = \mathbb{C}$.

Or from the very recent work [8]:

> The aim of this essay is to shine some light on this unexpected result ($\det(S) = +1$) from various angles, hoping to demystify it to some degree.

Our purpose here is to give a straightforward matrix theoretical proof that $\det S = +1$ when S is symplectic, but in the process, we give new information about the patterns of linearly independent rows and columns among the blocks of a symplectic matrix S. We refer to these relationships as *the complementary bases theorem*. The essential ingredients of our work are very simple: basic properties of Schur complements and the special block structure of the inverse of a symplectic matrix. One essential difference between the proof we present for $\det S = +1$ and those presented in [2,8,9], is that the new proof looks at the elements of the matrix, while the others pay attention to factorization properties of the symplectic group, or to other more abstract properties of symplectic matrices.

2. Preliminary results

In this section, we summarize some well-known basic results that will be used later on. Moreover, an extremely elementary proof of the determinant of certain symplectic matrices being one is presented. Unfortunately, this proof is not valid for an arbitrary symplectic matrix, but it is one of the fundamental ideas in our work.

Recall [5, Section 0.7.3, p. 18] that if an m-by-m *invertible matrix* A is partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

in which A_{11} *is nonsingular*, and if

$$A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

is partitioned conformally, we then have that B_{22} is nonsingular, and

$$B_{22}^{-1} = A_{22} - A_{21}A_{11}^{-1}A_{12},$$

i.e., B_{22}^{-1} is the Schur complement of A_{11} in A. Therefore

$$\det A = \det A_{11} \det[A_{22} - A_{21}A_{11}^{-1}A_{12}] = \frac{\det A_{11}}{\det B_{22}}. \tag{1}$$

On the other hand, any $2n$-by-$2n$ matrix is naturally thought of in partitioned form

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \tag{2}$$

in which S_{11} is n-by-n and we shall fix and refer to this partition throughout. A matrix S so partitioned is symplectic *if and only if*

$$S^{-1} = \begin{bmatrix} S_{22}^{\mathrm{T}} & -S_{12}^{\mathrm{T}} \\ -S_{21}^{\mathrm{T}} & S_{11}^{\mathrm{T}} \end{bmatrix}. \tag{3}$$

This is easily proven: Definition 1.1 implies that S is symplectic if and only if $S^{-1} = J^{-1}S^{\mathrm{T}}J = -JS^{\mathrm{T}}J$. Then some trivial block manipulations lead to the result above.

Applying the determinantal formula (1), we quickly obtain that if a symplectic matrix S has the block S_{11} *nonsingular* then

$$\det S = \frac{\det S_{11}}{\det S_{11}^{\mathrm{T}}} = \frac{\det S_{11}}{\det S_{11}} = 1.$$

The fact that $\det S = +1$ may also be easily obtained whenever at least one of the four blocks in (2) is nonsingular: simply perform some row and/or column interchanges in (2) to move the nonsingular block to the $(1, 1)$-position, and follow the track of these interchanges in the block expression of the inverse (3). Finally, (1) is applied to the resulting matrix. Let us explain with more detail this process in the specific case of S_{12} being nonsingular and S_{11} being singular. Let P be the permutation matrix that interchanges in S the jth column of S_{11} with the jth column of S_{12} for all $j \in \{1, \ldots, n\}$. Thus SP has S_{12} as its $(1, 1)$-block, and $(SP)^{-1} = P^{\mathrm{T}} S^{-1}$ has $-S_{12}^{\mathrm{T}}$ as its $(2, 2)$-block, because P^{T} interchanges the corresponding rows of S^{-1}. Eq. (1) implies that $\det(SP) = \det S_{12}/\det(-S_{12}) = (-1)^n$. Our claim follows from the fact that $\det(SP) = (-1)^n \det S$. The case S_{21} being nonsingular can be dealt with in the same way, interchanging rows of S instead of columns. Finally, if one wants to move S_{22} to the $(1, 1)$-block position, both rows and columns have to be interchanged. This is the simplest case because no sign changes appear.

Unfortunately, the following example shows that there exist symplectic matrices having the four blocks singular:

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right].$$

In the next section, the notion of complementary bases is introduced. This idea allows us to relax the restriction of at least one of the blocks being nonsingular in the above proof, making it entirely general.

To finish this section, let us comment that for conjugate symplectic matrices, i.e., those fulfilling $S^* J S = J$, the inverse of S has the structure appearing in (3), but replacing transpose by conjugate transpose matrices. Thus, in this case, (1) leads to $\det S = \frac{\det S_{11}}{\det S_{11}^*}$, which is not necessarily +1, but necessarily on the unit circle.

3. The complementary bases theorem

The main new result in this work is easily described in partitioned form in the next theorem. In this theorem, $|\alpha|$ denotes the cardinality of a set α. Moreover the binary variables p and q can take as values 1 or 2, and p' and q' denote, respectively, the complementary variables of p and q.

Theorem 3.1. *Let S be a $2n$-by-$2n$ symplectic matrix partitioned as in* (2). *Suppose that rank* $(S_{pq}) = k$, $p, q \in \{1, 2\}$, *and that the rows* (*columns*) *of S_{pq} indexed by* α, $\alpha \subseteq \{1, \ldots, n\}$ *and* $|\alpha| = k$, *are linearly independent. Then the rows* (*columns*) *of* $S_{p'q}$ $\left(S_{pq'}\right)$ *indexed by* α', *the complement of* α, *together with the rows* (*columns*) α *of* S_{pq} *constitute a basis of* $\mathbb{K}^n$, *i.e., they constitute a nonsingular n-by-n matrix.*

The fact that certain rows (columns) from one block, together with complementary rows (columns) from another constitute a basis is the motivation for the name we have chosen. In fact, Theorem 3.1 gathers eight different results: one for rows and the other for columns for each of the four blocks. As an specific illustration of the complementary bases result, let us consider a 10×10 symplectic matrix with rank $S_{11} = 2$, $S_{11} = [a_1, a_2, a_3, a_4, a_5]$, and $S_{12} = [b_1, b_2, b_3, b_4, b_5]$, where a_i and b_j denote columns. Assume that $\{a_1, a_4\}$ is a linearly independent set of columns, then the complementary bases result states that the 5×5 matrix $[a_1, b_2, b_3, a_4, b_5]$ is nonsingular.

It should be noticed that S is nonsingular. Therefore, the n-by-$2n$ matrix $[S_{11}\ S_{12}]$ has rank equal to n, and also has n linearly independent columns. The complementary bases theorem specifies, for symplectic matrices, a very particular set of n independent columns by using complementarity. This phenomenon need not occur within a general $2n$-by-$2n$ partitioned nonsingular matrix, as the reader can easily check by devising some examples.

Proof of Theorem 3.1. We only need to prove the theorem in the column case $p = 1$ and $q = 1$. This result applied on the symplectic matrix SJ proves the column case $p = 1$ and $q = 2$. Once these two cases are proved, the other six cases follow from (2) and (3), and the fact that S^{-1}, S^{T}, and $S^{-\mathrm{T}}$ are, all of them, symplectic matrices.

We begin by selecting an n-by-n permutation matrix P to move the maximal linearly independent set of columns of S_{11} indexed by α to the first k positions. This is accomplished by multiplying S on the right by the symplectic matrix $P \oplus P$ to get the symplectic matrix S':

$$S' := S \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} = \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix}. \tag{4}$$

Let Y be the n-by-n nonsingular matrix such that YS'_{11} is a row reduced echelon matrix, and let us multiply S' on the left by the symplectic matrix $Y \oplus Y^{-\mathrm{T}}$ to get the symplectic matrix S'':

$$S'' := \begin{bmatrix} Y & 0 \\ 0 & Y^{-\mathrm{T}} \end{bmatrix} \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix} = \begin{bmatrix} S''_{11} & S''_{12} \\ S''_{21} & S''_{22} \end{bmatrix},$$

where

$$S''_{11} = \begin{bmatrix} I_k & Z \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad S''_{12} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{5}$$

are partitioned conformally. Obviously, the multiplication above does not change the relationships of linear independence among the columns of $[S'_{11} S'_{12}]$, therefore to prove the complementary bases theorem is equivalent to prove that the $(n-k)$-by-$(n-k)$ matrix X_{22} is nonsingular. Bearing in mind that S'' is symplectic, its inverse has the block structure appearing in (3), then

$$0 = (S''S''^{-1})_{12} = -S''_{11}S''^{\mathrm{T}}_{12} + S''_{12}S''^{\mathrm{T}}_{11},$$

and substituting (5), we get

$$\left[\begin{array}{c|c} X_{11} - X_{11}^{\mathrm{T}} + X_{12}Z^{\mathrm{T}} - ZX_{12}^{\mathrm{T}} & -(X_{21}^{\mathrm{T}} + ZX_{22}^{\mathrm{T}}) \\ \hline X_{21} + X_{22}Z^{\mathrm{T}} & 0 \end{array}\right] = 0.$$

This implies

$$X_{21} = -X_{22}Z^{\mathrm{T}}.$$

Any subset of rows of S'' is linearly independent because S'' is nonsingular. In particular all the rows of the submatrix $[0\ 0\ X_{21}\ X_{22}]$ are linearly independent, thus

$$n - k = \operatorname{rank}[X_{21}\ X_{22}] = \operatorname{rank} X_{22}[-Z^{\mathrm{T}}\ I] = \operatorname{rank} X_{22}. \qquad \square$$

An immediate corollary of the complementary bases theorem gives information on the patterns of zeros compatible with the symplectic structure:

Corollary 3.2. *Let S be a* $2n$*-by-*$2n$ *symplectic matrix partitioned as in* (2). *Suppose that the rows* (*columns*) *of* S_{pq}, $p, q \in \{1, 2\}$, *indexed by* β, $\beta \subseteq \{1, \ldots, n\}$, *are zero. Then the rows* (*columns*) *of* $S_{p'q}$ $\left(S_{pq'}\right)$ *indexed by* β *are linearly independent.*

Proof. Every set of indices α considered in Theorem 3.1 is included in β'. Then β is included in α'. $\square$

It should be noticed that Theorem 3.1 and Corollary 3.2 are valid for conjugate symplectic matrices.

The statement of Theorem 3.1 suggests the following natural question: may the complementary bases theorem be extended to rows (columns) of S_{pq} and rows (columns) of $S_{pq'}$ $\left(S_{p'q}\right)$? Notice, that this would mean eight additional results. However, the following example shows that this extension is not true. Let us consider the symplectic matrix

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ \hline 0 & -1 & 1 & 0 \\ 1 & 2 & 1 & -1 \end{array}\right].$$

The rank of the $(1, 1)$-block is obviously 1 and its second row constitutes a maximal linearly independent set of rows. However, the second row of the $(1, 1)$-block together with the first row of the $(1, 2)$-block do not constitute a nonsingular matrix.

4. A proof that the determinant of a symplectic matrix is one

In this last section we prove, as a corollary of Theorem 3.1, the classical result which has motivated our work:

Theorem 4.1. *Let S be a* $2n$*-by-*$2n$ *symplectic matrix with entries in any field* $\mathbb{K}$. *Then* $\det S = +1$.

Proof. Let us consider again the partition (2) and assume that $\operatorname{rank}(S_{11}) = k$. Let us select as in (4) an n-by-n permutation matrix P to move a maximal linearly independent set of columns of S_{11} to the first k positions. Notice that $S' = S(P \oplus P)$ is symplectic and that $\det S' = \det S$, because $\det(P \oplus P) = (\det P)^2 = 1$. From now on, we will focus on the matrix S'.

Let us consider the following partition of the two upper blocks of S':

$$\left[\, S'_{11} \,\middle|\, S'_{12} \right] = [E_1 E_2 | F_1 F_2],$$

where E_1 and F_1 are n-by-k matrices, and E_2 and F_2 are n-by-$(n - k)$ matrices. Then, Theorem 3.1 implies that the n-by-n matrix $[E_1\ F_2]$ is nonsingular.

Let Π_j be the $2n$-by-$2n$ matrix that multiply by -1 the column j of S'_{11}, and then interchanges this column with the column j of S'_{12}. Therefore, Π_j has determinant +1 and has the following block structure

$$\Pi_j = \left[\begin{array}{c|c} Q_1 & -Q_2 \\ \hline Q_2 & Q_1 \end{array}\right],$$

where Q_1 is an n-by-n diagonal matrix with ones on the diagonal, except for a zero in the (j, j) position; and Q_2 has all the entries zero, except +1 in the (j, j) position. It can be trivially checked that Π_j is symplectic. Then

$$S'\Pi_{k+1}\cdots\Pi_n = \left[\begin{array}{c|c} E_1F_2 & * \\ \hline * & * \end{array}\right]$$

is also symplectic, and, taking into account the structure of the inverse of a symplectic matrix (3),

$$(S'\Pi_{k+1}\cdots\Pi_n)^{-1} = \left[\begin{array}{c|c} * & * \\ \hline * & \begin{matrix} E_1^{\mathrm{T}} \\ F_2^{\mathrm{T}} \end{matrix} \end{array}\right].$$

Now, the determinantal formula (1) is applied to the matrix $S'\Pi_{k+1}\cdots\Pi_n$:

$$\det S'\Pi_{k+1}\cdots\Pi_n = \frac{\det[E_1F_2]}{\det\begin{bmatrix} E_1^{\mathrm{T}} \\ F_2^{\mathrm{T}} \end{bmatrix}} = \frac{\det[E_1F_2]}{\det[E_1F_2]^{\mathrm{T}}} = 1.$$

Finally, $\det S'\Pi_{k+1}\cdots\Pi_n = \det S' = \det S$. Then

$$\det S = 1. \qquad \square$$

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